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THE USE OF GENERALIZED FUNCTIONS IN THE PROBLEM OF ELASTIC OSCILLATIONS OF A COMPOSITE ROD†

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The motion of a multicomponent anthropomorphic model is described by the equation of degenerate transverse oscillations of a one-dimensional system of rods with distributed mass and bending stiffness parameters and with concentrated inclusions into these parameters. The concentrated inclusions can be introduced into the analytic expression for the distributed parameters using impulse functions of the first and higher orders (the delta-function and its derivatives). The solution of the equation is found by the operational method. Krylov's generalized functions make it possible to obtain an analytic solution of the equation, which can be realized on a computer, and is written for the whole multicomponent structure at once in the framework of the direct and inverse problems of mechanics. Some optimization problems for a model of the motion of a sportsman are solved in the framework of the direct problem of mechanics.

1. FORMULATION OF THE PROBLEM AND FUNDAMENTAL EQUATIONS

CONSIDER a mechanical model of the human body undergoing oscillatory motion like a one-dimensional system of rods whose parameters (the density, the moment of inertia of the transverse cross-section of a rod about an axis perpendicular to the plane of oscillations, and the modulus of elasticity) correspond to the parameters of the body. The parts are connected by elastic joints. The centre of mass of each of the parts lies on the straight line connecting the centres of rotation of the neighbouring joints. The system moves in the plane of the oscillations. The motion of an athlete throwing a javelin serves as a model.

As a mathematical model of the process we consider the equation of forced transverse oscillations of a system of rods with distributed parameters and concentrated insertions into these parameters. Agashin [1] was the first to propose such a formulation of the problem of modelling the motion of the human body.

We will restrict ourselves to the modelling of concentrated inclusions into the distributed mass and bending stiffness parameters of the rod, which model joints with elastic constraints. Concentrated inclusions can be introduced into the analytic expression for the distributed parameters by means of the delta-function and its derivatives. In order to model the process, we use the theory of Schwartz–Sobolev generalized functions [2–4].

The equation for the transverse oscillations of a one-dimensional system of rods (when there are no longitudinal forces) has the form

$$\left(1+\chi\frac{\partial}{\partial t}\right)\frac{\partial^2}{\partial x^2}\left[EJ(x)\frac{\partial^2 U(x,t)}{\partial x^2}\right]+ \\ +m(x)\left[\frac{\partial^2 U(x,t)}{\partial t^2}+\zeta\frac{\partial U(x,t)}{\partial t}\right]=Q(x,t) \tag{1.1}$$

$$0\leqslant x\leqslant L,\quad t\in[t_0,t_1]$$

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where $U(x, t)$ is the equation of the elastic line of the system of rods, $m(x)$ and $EJ(x)$ are generalized functions representing the distributed properties as well as the concentrated inclusions into the linear mass and bending stiffness (E is the modulus of elasticity and J is the moment of inertia of the transverse cross-section of the rod about an axis perpendicular to the plane of oscillations), and $Q(x, t)$ is the distributed transverse force.

Since the dimensions of the cross-section are small compared with the length of the rod, we shall neglect the moment of inertia of the system. This idealization is justified for low oscillation frequencies in the framework of the problem under consideration [5].

The linear differential equation (1.1) with singular coefficients describes the transverse oscillations of the multicomponent system of rods to be modelled. It takes into account the internal forces of friction, which are proportional to the rate of change of the elastic restoring force (the Voigt hypothesis), and the external viscous resistance forces, which are proportional to the rate of transverse displacement of the points of the system of rods [2, 3, 6].

To simplify the solution, we will assume that the external viscous resistance forces are proportional to the linear mass of the system of rods. The assumption that the linear mass describes the configuration of the system can sometimes be justified [2, 3], e.g. in the case of a rod with a rectangular cross-section with a side of constant length parallel to the plane of oscillations. This being the case, if concentrated masses are present, then the concentrated viscous resistance forces with the same coefficient of viscosity ζ will be applied to them.

The linearity of the resistance forces makes it possible to integrate the equations of motion of complex systems. For a bounded system it is customary to use the oscillatory approach to the solution of the equation, i.e. the Bernoulli method. Any motion is then considered as a superposition of the characteristic oscillations of the system.

For unsteady modes of forced oscillations generated by dynamic sources (forces), the solution can be found by means of the representation as a superposition of some steady states of the system. Under certain boundary conditions, the forms of free oscillations of the rod with suitable parameters can be determined in advance irrespective of the specific load in the problem in question.

We state the following boundary conditions at the ends of the system of rods:

$$\partial^2 U(0, t)/\partial x^2 = 0, \quad \partial^3 U(0, t)/\partial x^3 = 0$$

$$U(L, T) = 0, \quad \partial^2 U(L, t)/\partial x^2 = 0$$

(The boundary conditions for the bending moment and transverse force are stated at the free end $x = 0$, while the boundary conditions for the deflection and bending moment are stated at the end $x = L$ with hinged support.)

We will seek a solution of the equation of motion as the sum

$$U(x, t) = \sum_{k=1}^{\infty} X_k(x) T_k(t) \tag{1.2}$$

where $X_k(x)$ is the kth natural mode of oscillation and $T_k(t)$ is the dynamic increment coefficient for the kth characteristic oscillation mode, where $k = 1, 2, \ldots$.

Based on the superposition principle, this method is justified only for linear systems, and, in principle, it introduces an error if the number of terms that are taken into account are restricted.

The fundamental functions $X_k(x)$ can be constructed using their orthogonality with weight $m(x)$, i.e.

$$\int_0^L m(x) X_k(x) X_n(x)\, dx = 0, \quad k \neq n$$

This property enables one to use the forms of characteristic oscillations as kernels of an integral transform, which makes it possible to obtain a solution for unsteady modes of oscillation (the unstable problem) as a series in terms of the eigenfunctions of the system [2].

As a rule, the intensity of the perturbation load can also be expanded in terms of the fundamental functions $X_k(x)$:

$$Q(x,t) = \sum_{k=1}^{\infty} a_k(t)\, m(x)\, X_k(x)$$

On separating the variables, Eq. (1.1) becomes the system of ordinary differential equations

$$(EJ(x)X_k''(x))'' - m(x)\nu_k^2 X_k(x) = 0$$

$$T_k''(t) + 2H_k T_k'(t) + \nu_k^2 T_k(t) = a_k(t), \quad H_k = {}^1/_2(\zeta + \chi\nu_k^2) \tag{1.3}$$

where ν_k is the characteristic frequency of the kth mode of oscillation, the first equation of the system describing free oscillations. We arrive at the simple problem of determining the scale of the oscillation modes considered only (i.e. the rate of damping of the oscillations), which can be solved taking energy dissipation into account. This approach, which rests on the specification of the damping coefficients applying to the characteristic modes of the system without damping, was adopted in [5, 7, 8], and is referred to as "damping over the modes of oscillation" in the literature.

2. A FORMAL SCHEME FOR CONSTRUCTING THE MAIN SOLUTION

Consider the solution of the first equation of system (1.3) under the assumption that the coefficients $m(x)$ and $EJ(x)$ are singular. The delta-function enables one to carry over the notions of mass per unit length and bending stiffness to the case of a discrete distribution [2, 3, 9, 10].

We will assume that there are a number of concentrated masses M_i at x_i ($i = 1, 3, 5, \ldots, 2n+1$) rigidly attached to a rod with constant linear mass m_0. For a system with constant stiffness ($EJ(x) = EJ_0 = \text{const}$), the distributed properties of the mass per unit length can be expressed by the equation

$$m(x) = m_0 + \sum_{i=1}^{2n+1} \frac{1+(-1)^{i+1}}{2} M_i \sigma_1(x - x_i)$$

($\sigma_1(x) = \sigma_0'(x)$ being a delta-function). Substituting this expression into the first equation of system (1.3) and using the continuity of the natural modes of oscillation, we obtain

$$X_k^{\mathrm{IV}}(x) - K_k^4 X_k(x) = \sum_{i=1}^{2n+1} \frac{1+(-1)^{i+1}}{2} \frac{M_i \nu_k^2}{EJ_0} X_k(x_i)\, \sigma_1(x - x_i) \tag{2.1}$$

$$K_k^4 = m_0\nu_k^2/(EJ_0)$$

Concentrated inclusions into the bending stiffness serve as a model of hinged elastic joints between the parts of the rod system at x_i ($i = 2, 4, \ldots, 2n$), i.e. there are no concentrated masses at the locations of the joints. The system is stable because all the joints are elastic. Moreover,

$$X_k(x) = X_k^0(x) + \sum_{i=1}^{2n+1} \frac{1+(-1)^i}{2} (X_k'(x_i - 0) - X_k'(x_i + 0)) \times$$
$$\times (x - x_i)\, \sigma_0(x - x_i) \tag{2.2}$$

where $X_k^0(x)$ is a continuous function of x with continuous first-order derivative, $X_k'(x_i - 0) - X_k'(x_i + 0) = \Delta X_k'(x_i)$ is the relative angle of rotation of the cross-section on the right- and left-hand sides of a joint, and $(x - x_i)\sigma_0(x - x_i)$ is the elementary linear spline [a polygonal (dashed) line]. The first equation of system (1.3) can be written as the system

$$X_k''(x) = (J_0/J(x))Z_k(x), \quad EJ_0 Z_k''(x) = m_0\nu_k^2 X_k(x) \tag{2.3}$$

where $Z_k(x)$ is the normalized "scaled" bending moment, which is a continuously differentiable function ($Z_k(x) \in C^1([0, L])$), and which must not be applied to the cross-section $x = x_i$, but only above or below the joint. In the case of an elastic joint

$$Z_k(x_i - 0) = K_{M_i}(EJ_0)^{-1} \Delta X_k'(x_i)$$

where K_{m_i} is the elastic constant of the joint.

Differentiating (2.2) twice with respect to x, we substitute the result into the first equation of system (2.3) and, using the equalities $(X_k^0(x))'' = Z_k(x)$ and $J(x) = J_0$ for $x \neq x_i$, we obtain

$$\frac{J_0}{J(x)} = 1 + \sum_{i=1}^{2n+1} \frac{1 + (-1)^i}{2} \frac{EJ_0}{K_{M_i}} \sigma_1(x - x_i) \tag{2.4}$$

Hence we find the final expression

$$X_k^{IV}(x) - K_k^4 X_k(x) = \sum_{i=1}^{2n+1} \frac{1 + (-1)^i}{2} \frac{EJ_0}{K_{M_i}} Z_k(x_i - 0) \sigma_3(x - x_i) \tag{2.5}$$

for the equation of free oscillation in the case of the impulse inclusions (2.4) into the compliance.

The motion of the system of rods can be described by linear equations. Thus we can use the superposition method to write down the general equation of free transverse oscillations of the system with distributed mass and stiffness parameters and concentrated inclusions into these parameters. We find that

$$\begin{aligned} X_k^{IV}(x) - K_k^4 X_k(x) = \sum_{i=1}^{2n+1} \Bigg[& \frac{1 + (-1)^{i+1}}{2} \frac{M_i \nu_k^2}{EJ_0} X_k(x_i) \sigma_1(x - x_i) + \\ & + \frac{1 + (-1)^i}{2} \frac{EJ_0}{K_{M_i}} Z_k(x_i - 0) \sigma_3(x - x_i) \Bigg] \end{aligned} \tag{2.6}$$

3. THE CHARACTERISTIC EQUATION AND ITS SOLUTION

We shall use the operational method to find the solution of the differential equation (2.6) with constant coefficients and singular right-hand side. By means of the formula

$$\bar{X}_k(p) = p \int_0^\infty e^{-pt} X_k(t)\, dt \tag{3.1}$$

we transform each of the terms forming the left- and right-hand sides of the differential equation into a function of a new variable p using the Carson–Heaviside transformation [11]. The transformation turns the differential equation (2.6) into an algebraic equation. The intermediate parameters $X_k(x_i)$ and $Z_k(x_i - 0)$ contained in the solution of the equation can be eliminated using recurrent relations. The final solution of the equation of free oscillation of the rod system with concentrated inclusions into the mass and bending stiffness of the rod can be written in the form

$$\begin{aligned} X_k(x) = \sum_{r=0}^{3} X_k^{(r)}(0) \Big\{ Y_r(x) + \sum_{i=1}^{2n+1} [& A_{ir} Y_3(x - x_i) \sigma_0(x - x_i) + \\ & + B_{ir} Y_1(x - x_i) \sigma_0(x - x_i)] \Big\} \end{aligned} \tag{3.2}$$

$$A_{ir} = \frac{1 + (-1)^{i+1}}{2} \frac{M_i \nu_k^2}{EJ_0} \Big\{ Y_r(x_i) + \sum_{j=1}^{i-1} [A_{jr} Y_3(x_i - x_j) + B_{jr} Y_1(x_i - x_j)] \Big\}$$

$$B_{ir} = \frac{1 + (-1)^i}{2} \frac{EJ_0}{K_{M_i}} \Big\{ Y_r''(x_i) + \sum_{j=1}^{i-1} [A_{jr} Y_1(x_i - x_j) + K_k^4 B_{jr} Y_3(x_i - x_j)] \Big\}$$

$$A_{1r} = M_1 \nu_k^2 (EJ_0)^{-1} Y_r(x_1), \ B_{1r} = 0$$

In the final analysis, the solution can be expressed in terms of the initial parameters and some influence functions only, namely, the Krylov functions [2, 3, 8]

$$Y_0(x) = \frac{1}{2}(\operatorname{ch} K_k x + \cos K_k x), \qquad Y_2(x) = \frac{1}{2K_k^2}(\operatorname{ch} K_k x - \cos K_k x)$$

$$Y_1(x) = \frac{1}{2K_k}(\operatorname{sh} K_k x + \sin K_k x), \qquad Y_3(x) = \frac{1}{2K_k^3}(\operatorname{sh} K_k x - \sin K_k x)$$

Rewriting the boundary conditions for the natural modes of oscillations $X_k(x)$ and substituting them into (3.2), we obtain the characteristic equation for K_k. Solving this transcendental equation on a computer, we can obtain the characteristic frequencies of the transverse oscillations of the system, as well as the modes of oscillation corresponding to them.

4. SOLUTION OF THE DIRECT PROBLEM OF MECHANICS

The problem of determining the reaction of the system subject to various external loads can be reduced to solving a linear differential equation, namely, the second equation of system (1.3). We can find the solution using the Duhamel integral [3, 5, 9]

$$T_k(t) = \left(T_{k_0}\cos(\psi_k t) + \frac{T_{k_0}^{\cdot} + H_k T_{k_0}}{\psi_k}\sin(\psi_k t)\right)\exp(-H_k t) +$$

$$+ \frac{1}{\psi_k}\int_0^t a_k(\tau)\exp(-H_k(t-\tau))\sin(\psi_k^{\cdot}(t-\tau))\,d\tau, \qquad \psi_k = (\nu_k^2 - H_k^2)^{1/2}$$

where $T_{k_0} = T_k(0)$ and $T^{\cdot}_{k_0} = T_k^{\cdot}(0)$ are the initial conditions.

When there are friction forces in the system, critical damping will take place for $H_k = \nu_k$ and, beginning from one term of the series, all the subject terms will describe a non-periodic damped motion.

5. BASIC APPLICATIONS

The values of the parameters of the system were chosen from the data in the literature. The experimental data for the real motion of an athlete during the final phase of throwing a javelin were obtained by a high-speed camera.

The solutions of the inverse and direct problems of mechanics in closed form are of special interest in applications. The proposed model enables one to obtain an analytic solution of the equation for the whole system at once suitable for computer programming. As the objective function of the process under investigation we shall take the speed of the hand holding the javelin at the moment of releasing the projectile.

The motion of the upper part of the body was modelled and suitable initial data for the equation were also given. The first five characteristic frequencies of the kinematic chain subject to modelling were computed: 1.98; 6.86; 11.91; 19.37; 44.32 Hz. The modes of oscillation of the system corresponding to these frequencies are presented in Fig. 1 (they are numbered in order of increasing frequency). The bending stiffness of the parts of the body exceeds that of the joints by several orders of magnitude. The natural modes of oscillation are therefore close to a polygonal line. The oscillations due to the active forces of the athlete play the role of forced oscillations of the kinematic chain.

In practice the model can be realized in the framework of the direct and inverse problems of mechanics. We can obtain the solution of the inverse problem of mechanics by solving the second equation of system (1.3) with respect to $a_k(t)$, i.e. expanding the motion in terms of the natural modes. We find the numerical values of the dynamic increment coefficient $T_k(t)$ for the natural modes of oscillation using the orthogonality of the fundamental functions $X_k(x)$. The model indicates that the motion being analysed can be constructed from the first three modes of oscillation.

It is interesting that the solution can be obtained within the framework of the direct problem of mechanics.

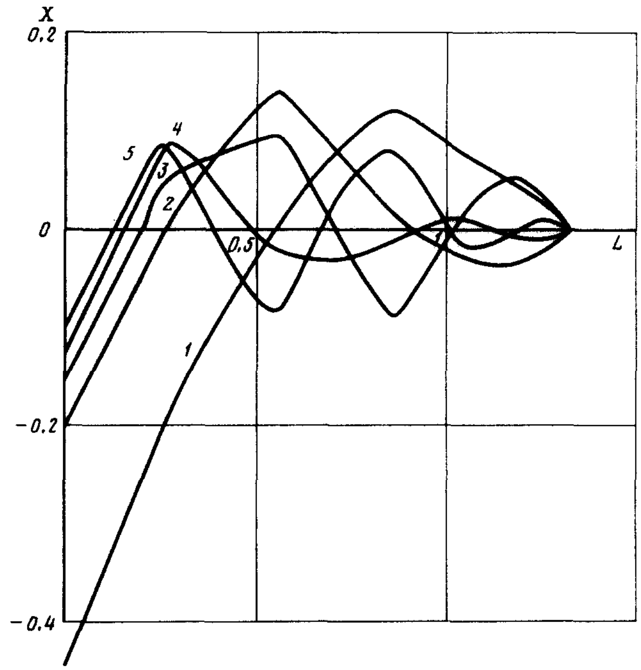

FIG. 1.

The control of the structure of the motion of the multicomponent system was considered as an example. The distribution of the time of attaining the extremum values of the control momenta for the parts of the human body served as a criterion for the effectiveness of the oscillatory motion being studied [12]. The optimality of the mechanisms of constructing the motion was investigated in a specific realization. The positions of the extrema of the control (muscular) forces on the arm (ΔT_1) and the body (ΔT_2) were simultaneously varied with time.

In Fig. 2 (the lines of constant time are numbered in order of decreasing speed: 30.62; 30.5; 30.4; 30.2; 30.0 m/s) we present the results of modelling the changes of the instants of application of the maximum values of the muscular forces on the arm and the body (the initial motion corresponds to the zero position on the time axis)

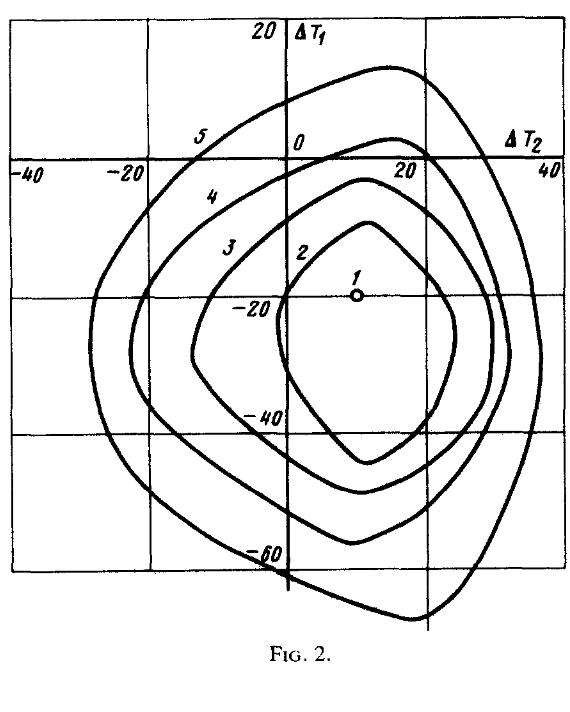

FIG. 2.

and finding on this basis the global maximum of the speed at the instant the javelin is released. In this way it is seen that the initial oscillatory motion of the thrower is not optimal (as far as the velocity criterion is concerned).

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